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CONTINUOUS SELECTIONS AVOIDING A SET

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Conditions are obtained under which a set-valued function $\varphi: X \to 2^Y$ has a continuous selection which avoids a set $E \subset Y$.

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Continuous selections

1. Introduction

The results of this paper are related to the following Theorems 1.1 and 1.2. All our maps—in particular, all our selections—are assumed to be *continuous.*

Theorem 1.1 [6, Theorem 1; 7, Theorem 3.2″]. *Let X be paracompact, Y a Banach space, and let $\varphi: X \to \mathscr{F}_c(Y)$ be l.s.c. Then φ has a selection.*[1]

Remark 1.1′. As observed in [7], Theorem 1.1 can be strengthened to conclude that φ has the *selection extension property*, that is, if $A \subset X$ is closed then every selection for $\varphi | A$ extends to a selection for φ. (It suffices to apply Theorem 1.1 to the l.s.c. function $\varphi_g: X \to \mathscr{F}_c(Y)$ defined by $\varphi_g(x) = \varphi(x)$ if $x \notin A$ and $\varphi_g(x) = \{g(x)\}$ if $x \in A$). A similar comment applies to most other selection theorems, including the following result.

Theorem 1.2 [8, Theorem 1.2]. *Let X be paracompact, Y completely metrizable, and $\varphi: X \to \mathscr{F}(Y)$ l.s.c. Suppose that* $\dim X \leq n+1$, *that $\varphi(x)$ is C^n for all $x \in X$, and that $\{\varphi(x): x \in X\}$ is* equi-LC^n. *Then φ has a selection.*[2]

[1] Letting $2^Y = \{E \subset Y: E \neq \emptyset\}$, we write $\mathscr{F}(Y) = \{E \in 2^Y: E \text{ closed}\}$ and $\mathscr{F}_c(Y) = \{E \in \mathscr{F}(Y): E \text{ convex}\}$. A function $\varphi: X \to 2^Y$ is *l.s.c.* (= *lower semi-continuous*) if $\{x \in X: \varphi(x) \cap V \neq \emptyset\}$ is open in X for every open V in Y. A *selection* for $\varphi: X \to 2^Y$ is a map $f: X \to Y$ such that $f(x) \in \varphi(x)$ for every $x \in X$.

[2] We assume $n \geq -1$. As usual, dim denotes the covering dimension. A space E is C^n if, for every $k \leq n$, every map from the k-sphere S^k into E is null-homotopic. A collection $\mathscr{E} \subset 2^Y$ is *equi-LC^n* if, for every $y \in \bigcup \mathscr{E}$, every neighborhood V of y in Y contains a neighborhood W of y in Y such that, for all $E \in \mathscr{E}$ and $k \leq n$, every map from S^k into $W \cap E$ is null-homotopic over $V \cap E$.

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The conditions on the sets $\varphi(x)$ in Theorem 1.2 are somewhat complicated, but they are purely topological and thus applicable in a wide variety of situations. By contrast, the condition in Theorem 1.1 is simple but rigid—without convexity, it cannot be applied at all. No purely topological conditions for selections with infinite-dimensional domain are known; in fact, Pixley [12] has constructed an example to show that a very natural (in view of Theorem 1.2) candidate for such conditions will not work. One of the results in this paper provides another such example (see Example 10.3); it is simpler than Pixley's, but the sets $\varphi(x)$—unlike Pixley's—are not compact.

We now turn to our principal topic, namely conditions under which the selection f in Theorem 1.1 can be chosen to *avoid* some subset E of Y, that is, so that $f(x) \notin E$ for all $x \in X$. We call such a selection *E-avoiding*; in the important special case where $E = \{0\}$, we call it *0-avoiding*. In general, the selection in Theorem 1.1 cannot be chosen to be 0-avoiding (even when $\varphi(x)\backslash\{0\} \neq \emptyset$ for all $x \in X$); see Examples 10.1 and 10.2. Nevertheless, the following positive result was recently obtained (under slightly stronger hypotheses) by Saint Raymond in [13, Lemma].[3]

Theorem 1.3. *Let X be paracompact, Y a Banach space, $\varphi : X \to \mathscr{F}_c(Y)$ l.s.c., and suppose that* $\dim X < \dim \varphi(x)$ *whenever $x \in X$ and $0 \in \varphi(x)$. Then φ has a 0-avoiding selection.*

The conclusion of Theorem 1.3 can easily be strengthened in two directions. First, as in Remark 1.1′, one may conclude that, if $A \subset X$ is closed, then every 0-avoiding selection for $\varphi | A$ extends to a 0-avoiding selection for φ. Second, given any map $g : X \to Y$, one can find a selection f for φ such that $f(x) \neq g(x)$ for all $x \in X$ (it suffices to use Theorem 1.3 to find a 0-avoiding selection f' for $\varphi - g$, and then to let $f = f' + g$); for a generalization, see Corollary 4.3.

The proof of Theorem 1.3 given in [13] (for compact metric X and assuming $0 \in \varphi(x)$ for all $x \in X$) uses Theorem 1.1 and proceeds by induction on the dimension of X. Alternatively, one can use Theorem 1.2 to give a fairly simple proof of a more general result in which 0 is replaced by a suitable closed subset of Y (see Theorem 2.2). Further generalizations are given in Sections 3 and 4. For example, Theorem 3.3 implies that the selection for φ can be chosen to avoid, not just one, but countably many suitable closed subsets of Y; this implies, in particular, that the selection in Theorem 1.3 can actually be chosen to avoid any given countable subset of Y.

We now consider a rather natural question: Does the conclusion of Theorem 1.3 remain true without any dimensional restrictions on X but assuming that $\dim \varphi(x) = \infty$ for all $x \in X$? As we shall see in Example 10.2, the answer, in general, is "no", even with X and every $\varphi(x)$ compact and with φ continuous[4] rather than just l.s.c. Nevertheless, we have the following positive result.

[3] I am grateful to B. Ricceri for calling Saint Raymond's result (which helps to answer a question of Ricceri) to my attention.

[4] i.e., continuous with respect to the Hausdorff metric on $\mathscr{F}(Y)$, defined by $\rho(E_1, E_2) = \inf\{\varepsilon > 0 : E_1 \subset B_\varepsilon(E_2), E_2 \subset B_\varepsilon(E_1)\}$. (While this may give infinite values for unbounded sets, it still yields a metrizable topology). Every continuous $\varphi : X \to \mathscr{F}(Y)$ is l.s.c.

Theorem 1.4. *Let X be a space, Y a Banach space, and $\varphi : X \to \mathscr{F}_c(Y)$ continuous with* $\dim \varphi(x) = \infty$ *for all* $x \in X$. *Suppose that, if* $x \in X$ *and* $0 \neq y \in \varphi(x)$, *then* $\|y\|^{-1} y \in \varphi(x)$. *Then φ has a 0-avoiding selection.*

The validity of Theorem 1.4 for arbitrary (rather than paracompact) spaces X is a pleasant by-product of the assumption that φ is continuous (rather than merely l.s.c.). Even if X is compact metric, however, the assumption that φ is continuous cannot be weakened to assuming that it is l.s.c. (see Example 10.4), and the last assumption on φ cannot be omitted (see Example 10.2). The following question remains open.

Question. Let X be paracompact, Y a Banach space, and $\varphi : X \to \mathscr{F}_c(Y)$ l.s.c. with each $\varphi(x)$ an infinite-dimensional linear subspace of Y. Must φ have a 0-avoiding selection?

Theorem 1.4 (or the closely related Theorem 7.1) can be used to prove the following result. (For $B = l_2$, a simple and direct proof of this result was obtained by my colleague Isaac Namioka, and it would be interesting to know whether such a proof also exists for an arbitrary infinite-dimensional Banach space B.)

Theorem 1.5. *Suppose B is an infinite-dimensional*[5] *Banach space, B^* its dual,* $S = \{x \in B : \|x\| = 1\}$ *and* $S^* = \{u \in B^* : \|u\| = 1\}$. *Then*:

(a) *There exists a map $f : S^* \to S$ such that, for all $u \in S^*$, $u(f(u)) = 0$ and $f(\lambda u) = f(u)$ whenever $|\lambda| = 1$.*

(b) *There exists a map $g : S \to S^*$ such that, for all $x \in S$, $[g(x)](x) = 0$ and $g(\lambda x) = g(x)$ whenever $|\lambda| = 1$.*

It should be observed that, by embedding B in B^{**}, assertion (b) of Theorem 1.5 follows immediately from (a).

The paper is arranged as follows. Sections 2–5 deal with finite-dimensional X (or subsets of X), while Sections 6–9 impose no dimensional restrictions on X. In Section 2 we prove a generalization of Theorem 1.3, and further generalizations are obtained in Sections 3 and 4. An open question is raised at the beginning of Section 4. Section 5 examines the 0-dimensional case. We then prove a general selection theorem for continuous φ in Section 6, and apply it in Section 7 to prove Theorem 1.4 and some related results. Theorem 1.5 is proved in Section 8, and some additional selection theorems are obtained in Section 9. Section 10, finally, is devoted to examples.

2. A generalization of Theorem 1.3

We will prove a generalization of Theorem 1.3 (see Theorem 2.2 below) with the aid of Theorem 1.2. To guarantee that the conditions of Theorem 1.2 are satisfied

[5] This result is false if B is finite-dimensional.

in our situation, we first need the following lemma. We denote the convex hull of a set E by conv E.

Lemma 2.1. *Let Y be a Banach space, $K \subset Y$ convex, and $E \subset Y$ closed. Define $L = \operatorname{conv}(E \cap K)$, and suppose $n \geq 0$ and* $\dim K > \dim L + n + 1$. *Then $K \setminus E$ is C^n.*

Proof. Let $k \leq n$. We must show that every map $f: S^k \to K \setminus E$ is null-homotopic. Since E is closed in Y, the map f is homotopic to a piecewise linear map $g: S^k \to K \setminus E$. Let $F = g(S^k)$. It will suffice to show that F is contractible in $K \setminus E$.

Clearly $F = \bigcup_{j=1}^{i} F_j$, with F_j convex and $\dim F_j \leq k$ for all j. For each j, let D_j be the affine subspace of Y spanned by L and F_j, and let $D = \bigcup_j D_j$. Let $m = \dim L$. Then $\dim D_j \leq m + n + 1$, and therefore $\dim D \leq m + n + 1$. Thus $D \not\supset K$, so we can pick $y \in K \setminus D$. Our choice of y implies that no line through y intersects both L and F, so F is linearly contractible to y in $K \setminus E$. □

It appears difficult to improve the inequality in Lemma 2.1. On the one hand, $n+1$ cannot be reduced to n, even when $n = 0$, as is seen by taking K to be an interval and E its midpoint. On the other hand, the inequality cannot be changed to $\dim K > \dim E + n + 1$, even when $n = 1$: Indeed, $\mathbb{R}^3$ has a compact subset E (Antoine's necklace) such that $\dim E = 0$ but $\mathbb{R}^3 \setminus E$ is not C^1 (see [4, pp. 177–178]).

The following theorem reduces to Theorem 1.3 when $E = \{0\}$.

Theorem 2.2. *Let X paracompact, Y a Banach space, and $\varphi: X \to \mathcal{F}_c(Y)$ l.s.c. Let $E \subset Y$ be closed, define $Z = \{x \in X: \varphi(x) \cap E \neq \emptyset\}$, and suppose that*

$$\dim X < \dim \varphi(x) - \dim \operatorname{conv}(\varphi(x) \cap E)\ [6]$$

for all $x \in Z$. Then φ has an E-avoiding selection.

Proof. Let $Y' = Y \setminus E$. Then Y' is open in Y and hence completely metrizable. Define $\psi: X \to \mathcal{F}(Y')$ by $\psi(x) = \varphi(x) \cap Y'$. We must show that ψ has a selection, which we do by checking that it satisfies the hypotheses of Theorem 1.2 with n defined by $\dim X = n + 1$.

First, ψ is l.s.c. because φ l.s.c. and Y' is open in Y. Next, $\psi(x)$ is C^n for all $x \in X$; this is clear when $x \notin Z$ (because then $\psi(x) = \varphi(x)$, so $\psi(x)$ is convex), and when $x \in Z$ it follows from Lemma 2.1 with $K = \varphi(x)$. To show, finally, that $\{\psi(x): x \in X\}$ is equi-LC^n, let $y \in Y'$ and let V be a neighborhood of y in Y'. Then y has a neighborhood $W \subset V$ in y' which is a convex subset of Y. But then $W \cap \psi(x) = W \cap \varphi(x)$ for all $x \in X$, so $W \cap \psi(x)$ is convex and thus C^n. □

The example following Lemma 2.1 implies that $\dim \operatorname{conv}(\varphi(x) \cap E)$ cannot be replaced by $\dim(\varphi(x) \cap E)$ or by $\dim E$ in Theorem 2.2, even when $\dim E = 0$.

[6] It is implicit in this inequality that $\dim X$ and $\dim \operatorname{conv}(\varphi(x) \cap E)$ are finite, but $\dim \varphi(x)$ may be infinite. A similar comment applies to results in Sections 3 and 4.

Example 10.6 shows that the assumption that E is closed in Y cannot be dropped, even when $\dim X = 0$ and $\varphi(x) \cap E$ is a singleton for every $x \in X$; however, it follows from Theorem 3.3 that it suffices for E to be an F_σ in Y. The question whether $\dim X$ can be replaced by $\dim Z$ will be considered in Section 4.

As in Remark 1.1′, Theorem 2.2 can be strengthened to conclude that, if $A \subset X$ is closed, then every E-avoiding selection for $\varphi \mid A$ extends to an E-avoiding selection for φ. If A is also a G_δ in X, then we can draw the following sharper conclusion.

Corollary 2.3. *Let X, Y, $\varphi : X \to \mathscr{F}_c(Y)$ and $E \subset Y$ be as in Theorem 2.2, and let A be a closed G_δ in X. Then every selection g for $\varphi \mid A$ extends to a selection f for φ such that $f(x) \notin E$ for every $x \in X \setminus A$.*

Proof. Since A is a closed G_δ in the normal space X, there is a map $u : X \to [0, 1]$ such that $A = u^{-1}(0)$. Extend g to a selection h for φ (using Remark 1.1′), and define $\psi : X \setminus A \to \mathscr{F}_c(Y)$ by $\psi(x) = (\varphi(x) \cap B_{u(x)}(h(x)))^-$. Then ψ is l.s.c., and $\dim \psi(x) = \dim \varphi(x)$ for every $x \in X \setminus A$ because $\psi(x) \subset \varphi(x)$ and $\psi(x)$ contains a non-empty, relatively open subset of $\varphi(x)$. Also $X \setminus A$ is paracompact because it is an F_σ in X. Hence ψ satisfies all the hypotheses of Theorem 2.2, and thus it has an E-avoiding selection f'. Now define $f : X \to Y$ by $f \mid A = g$ and $f \mid (X \setminus A) = f'$. Then f is continuous because $d(f(x), h(x)) \leq u(x)$ for all $x \in X \setminus A$, and hence f satisfies all our requirements. □

3. Some generalizations of Theorem 2.2

The following result reduces to Theorem 2.2 if $\psi(x) = E$ for all $x \in X$. We say that $f : X \to Y$ *avoids* $\psi : X \to \mathscr{F}(Y)$ if $f(x) \notin \psi(x)$ for all $x \in X$.

Theorem 3.1. *Let X be paracompact, Y a Banach space, and $\varphi : X \to \mathscr{F}_c(Y)$ l.s.c. Let $\psi : X \to \mathscr{F}(Y)$ be continuous*[7]*, define $Z = \{x \in X : \varphi(x) \cap \psi(x) \neq \emptyset\}$, and suppose that*

$$\dim X < \dim \varphi(x) - \dim \operatorname{conv}(\varphi(x) \cap \psi(x))$$

for all $x \in Z$. Then φ has a ψ-avoiding selection.

Proof. Metrize $\mathscr{F}(Y)$ with the Hausdorff metric. Let $D = \{(A, y) \in \mathscr{F}(Y) \times Y : y \in A\}$, and note that D is closed in $\mathscr{F}(Y) \times Y$. Embed $\mathscr{F}(Y)$ in a Banach space B, and pick a closed $E \subset B \times Y$ such that $E \cap (\mathscr{F}(Y) \times Y) = D$.

Observe that $B \times Y$ is also a Banach space, and define $\varphi : X \to \mathscr{F}_c(B \times Y)$ by $\varphi'(x) = \{\psi(x)\} \times \varphi(x)$. Then φ' is also l.s.c., and φ' and $E \subset B \times Y$ satisfy the assumptions of Theorem 2.2, so φ' has an E-avoiding selection f'. Now let $f = \pi_Y \circ f'$, where $\pi_Y : B \times Y \to Y$ is the projection. This f is a ψ-avoiding selection for φ. □

[7] See Footnote 4.

Theorem 3.1 remains true if, instead of assuming that $\psi: X \to \mathscr{F}(Y)$ and that ψ is continuous, one assumes only that $\psi: X \to \mathscr{P}(Y)$ [8] and that ψ has a closed graph in $X \times Y$. For metrizable X, the proof is similar to—and a bit simpler than—the proof of Theorem 3.1. If X is only paracompact, however, then the proof becomes more complicated, and details will appear elsewhere.[9]

Our next result shows that, in Theorem 3.1, there are 'many' ψ-avoiding selections for φ.

Lemma 3.2. *Let X, Y, $\varphi: X \to \mathscr{F}_c(Y)$ and $\psi: X \to \mathscr{F}(Y)$ be as in Theorem 3.1. Let S be the set of selections for φ, topologized with the fine topology.[10] Then S is Baire space, and $U = \{f \in S: f \text{ avoids } \psi\}$ is open and dense in S.*

Proof. Since Y is complete, it follows from [11, p. 297, Ex. 14] that $C(X, Y)$ is a Baire space in the fine topology, and so is every uniformly closed subset of $C(X, Y)$ such as S. Let us show that U is open and dense in S, using the notation of Footnote 10.

It is easy to see that U is open in S, for if $f \in U$ and if $\gamma \in \Gamma$ is defined by $\gamma(x) = d(f(x), \psi(x))$ then clearly $V_\gamma(f) \cap S \subset U$. [An alternative construction of γ, using only the closedness of the graph (rather than the continuity) of ψ: Choose an open cover (U_n) of X such that $d(f(x), \psi(x)) > 1/n$ for $x \in U_n$, let (p_n) be a locally finite partition of unity subordinated to (U_n), and define $\gamma(x) = \sum_n (1/n) p_n(x)$.] It remains to show that U is dense in S. So suppose $f \in S$ and $\gamma \in \Gamma$, and let us show that $V_\gamma(f) \cap U \neq \emptyset$. Define $\varphi': X \to \mathscr{F}_c(Y)$ by $\varphi'(x) = (\varphi(x) \cap B_{\gamma(x)/2}(f(x)))^-$. Then φ' is l.s.c., and $\varphi'(x) \subset \varphi(x)$ for all $x \in X$. Also $\dim \varphi'(x) = \dim \varphi(x)$ for all $x \in X$ because $\varphi'(x)$ has a non-empty interior relative to the convex set $\varphi(x)$. Since φ satisfies the assumptions of Theorem 3.1, it is now easy to see that φ' also satisfies them, so φ' has a ψ-avoiding selection g. Clearly $g \in V_\gamma(f) \cap U$. $\square$

We now use Lemma 3.2 to show that, in Theorem 3.1, the selection for φ can be chosen to avoid, not just one, but countably many ψ_i.

Theorem 3.3. *Let X be paracompact, Y a Banach space, and $\varphi: X \to \mathscr{F}_c(Y)$ l.s.c. Let $\psi_i: X \to \mathscr{F}(Y)$ $(i = 1, 2, \dots)$ be continuous, define $Z_i = \{x \in X: \varphi(x) \cap \psi_i(x) \neq \emptyset\}$, and suppose that*

$$\dim X < \dim \varphi(x) - \dim \operatorname{conv}(\varphi(x) \cap \psi_i(x))$$

for all $x \in Z_i$ and all i. Then φ has a selection which avoids every ψ_i.

[8] $\mathscr{P}(Y) = \{E: E \subset Y\}$.

[9] See also results in Section 5 and Theorem 9.1.

[10] The *fine topology* $\mathscr{T}$ on a function space $C(X, Y)$, with Y a metric space, is defined as follows [11, p. 285, Ex. 9]: Let Γ be the set of all continuous $\gamma: X \to \mathbb{R}^+$, and for each $f \in C(X, Y)$ and $\gamma \in \Gamma$ let $V_\gamma(f) = \{g \in C(X, Y): d(g(x), f(x)) < \gamma(x) \text{ for all } x \in X\}$. Then $\mathscr{T}$ is the topology on $C(X, Y)$ in which $\{V_\gamma(f): \gamma \in \Gamma\}$ in a base at f for every $f \in C(X, Y)$.

Proof. Immediate from Lemma 3.2. □

Theorem 3.3 appears to be new even when each ψ_i is single-valued and all the sets $\varphi(x)$ are equal to the same set $K \in \mathscr{F}_c(Y)$; note that, in this case, the inequality reduces to $\dim X < \dim K$. However, if we further assume that all the ψ_i coincide and that K is the n-cube, then we recapture a result of Holsztynski [3]; I am grateful to my colleague Jack Segal for this reference.

In conclusion, it should be noted that an analogue of Remark 1.1′ applies to all the results of this section, as well as Sections 4 and 5: For each of these results, if $A \subset X$ is closed, then every 'suitably avoiding' selection for $\varphi | A$ extends to a 'suitably avoiding' selection for φ. Moreover, just as in Corollary 2.3, if $A \subset X$ is a closed G_δ, then *every* selection for $\varphi | A$ extends to a selection for φ which is 'suitably avoiding' on $X \backslash A$.

4. Replacing dim X by dim Z

This section is concerned with the following question: Can Theorem 2.2 be strengthened by replacing $\dim X$ by $\dim Z$ or even by $\dim_X Z$?[11] That appears to be quite plausible, since any selection for φ in Theorem 2.2 automatically avoids E on $X \backslash Z$, and hence only Z and not all of X should play a role in the inequality. Unfortunately, I am unable to answer this question in full generality, but a positive answer can be given in several important special cases.

First, the answer is easily seen to be positive if Z is closed in X. Indeed, it then follows from Theorem 2.2 that $\varphi | Z$ has an E-avoiding selection g, and this g can be extended to a (necessarily E-avoiding) selection for φ by Remark 1.1′. Second, Theorem 4.1 below provides a positive answer if $\sup_{x \in X} \dim \operatorname{conv}(\varphi(x) \cap E) < \infty$, a condition which is, in particular, satisfied whenever $\dim \operatorname{conv} E < \infty$. A third condition for a positive answer is that $\dim \varphi(x) < \infty$ for all $x \in X$; see Corollary 4.5. Finally, the answer is positive if $\dim_X Z = 0$; see Theorem 5.7 in the next section.

Theorem 4.1. *Theorem* 2.2 *remains true with* $\dim X$ *replaced by* $\dim_X Z$, *provided* $\sup_{x \in X} \dim \operatorname{conv}(\varphi(x) \cap E) < \infty$.

The proof of Theorem 4.1 will consist of an inductive application of the following somewhat weaker result. We denote $\sup_{x \in X} \dim \operatorname{conv}(\varphi(x) \cap E)$ by m_φ.

Lemma 4.2. *Theorem* 2.2 *remains true if the inequality is replaced by* $\dim_X Z < \dim \varphi(x) - m_\varphi$.

Proof. Let $k = \dim_X Z + m_\varphi$, so $\dim \varphi(z) > k$ for every $z \in Z$. Fix $z \in Z$. Then $\dim \operatorname{conv} F_z > k$ for some finite $F_z \subset \varphi(z)$. By Remark 1.1′, there is a finite set S_z of

[11] $\dim_X Z$ denotes $\sup\{\dim A: A \subset Z, A \text{ closed in } X\}$. See [8].

selections for φ such that $F_z = \{f(z): f \in S_z\}$. Define $\psi_z: X \to \mathscr{F}_c(Y)$ by $\psi_z(x) = \operatorname{conv}\{f(x): f \in S_z\}$. Then ψ_z is both l.s.c. and u.s.c.[12] Let $U_z = \{x \in X: \dim \psi_z(x) > k\}$; then $z \in U_z$, and U_z is open in X because ψ_z is l.s.c.

Let g be any selection for φ, and let $U_g = \{x \in X: g(x) \notin E\}$. Then U_g is open in X and $U_g \supset X \backslash Z$. Let $\Lambda = Z \cup \{g\}$. Then $\{U_\lambda : \lambda \in \Lambda\}$ is an open cover of X, so it has a locally finite partition of unity $\{p_\lambda : \lambda \in \Lambda\}$ subordinated to it. Let $\psi_g(x) = \{g(x)\}$, and define $\psi: X \to \mathscr{F}_c(Y)$ by

$$\psi(x) = \sum_{\lambda \in \Lambda} p_\lambda(x)\psi_\lambda(x).$$

Then ψ is also l.s.c. and u.s.c., and $\psi(x) \subset \varphi(x)$ for every $x \in X$.

Let $A = \{x \in X: \psi(x) \cap E \neq \emptyset\}$. Then A is closed in X (because ψ is u.s.c.) and $A \subset Z$, so $\dim A \leq \dim_X Z$. If $x \in A$, then $p_g(x) \neq 1$ (for if $p_g(x) = 1$ then $\psi(x) = \{g(x)\}$ and $g(x) \notin E$ so that $x \notin A$), hence $p_{z_0}(x) > 0$ for some $z_0 \in Z$, and therefore $\dim \psi(x) \geq \dim \psi_{z_0}(x) > k$. Thus

$$\dim A \leq \dim_X Z = k - m_\varphi < \dim \psi(x) - \dim \operatorname{conv}(\psi(x) \cap E)$$

for every $x \in A$, so we can apply Theorem 2.2 to $\psi | A$ to obtain an E-avoiding selection g for $\psi | A$. By Remark 1.1′, we can extend g to a (necessarily E-avoiding) selection f for ψ, and this f is an E-avoiding selection for φ. □

Proof of Theorem 4.1. We proceed by induction on m_φ (as defined before Lemma 4.2). The theorem is clearly true if $m_\varphi = -1$, so let us assume it for $m_\varphi = n-1$ and prove it for $m_\varphi = n$.

Let $A = \{x \in X: \dim \varphi(x) \leq \dim_X Z + n\}$, and note that A is closed in X because φ is l.s.c. Let $\psi = \varphi | A$, and define m_ψ analogously to m_φ. Then $m_\psi \leq n-1$ by the inequality in Theorem 2.2 which we are assuming, so we can apply our inductive hypothesis to ψ to obtain an E-avoiding selection for ψ. Define $\varphi': X \to \mathscr{F}_c(Y)$ by $\varphi'(x) = \varphi(x)$ if $x \in X \backslash A$ and $\varphi'(x) = \{g(x)\}$ if $x \in A$; since A is closed in X, this φ' is also l.s.c. Define $Z' \subset Z$ by $Z' = \{x \in X: \varphi'(x) \cap E \neq \emptyset\}$, and define $m_{\varphi'}$ analogously to m_φ. Now if $x \in Z'$, then $x \notin A$, so $\dim \varphi'(x) = \dim \varphi(x) > \dim_X Z + n = \dim_X Z + m_\varphi \geq \dim_X Z' + m_{\varphi'}$. Hence φ' satisfies the assumptions of Lemma 4.2, so φ' has an E-avoiding selection f, and this f is an E-avoiding selection for φ. □

Just as Theorems 3.1 and 3.3 followed from Theorem 2.2, the obvious modifications of these results follow from Theorem 4.1. Thus Theorem 3.3 remains true with $\dim X$ replaced by $\dim_X Z_i$, provided $\sup_{x \in X} \dim \operatorname{conv}(\varphi(x) \cap \psi_i(x)) < \infty$ for all i. This yields the following corollary as a very special case.

Corollary 4.3. *Let X be paracompact, Y a Banach space, $\varphi: X \to \mathscr{F}_c(Y)$ l.s.c., and $g_i: X \to Y$ continuous $(i = 1, 2, \ldots)$. Define $Z_i = \{x \in X: g_i(x) \in \varphi(x)\}$, and suppose*

[12] $\psi: X \to 2^Y$ is *u.s.c.* (= *upper semi-continuous*) if $\{x \in X: \psi(x) \subset V\}$ is open in X for every open V in Y.

that $\dim_X Z_i < \dim \varphi(x)$ *for all* $x \in Z_i$ *and all* i. *Then* φ *has a selection* f *such that* $f(x) \neq g_i(x)$ *for all* $x \in X$ *and all* i.

Our next result, in which $\dim_X Z$ is not required to be finite, is another application of Theorem 4.1.

Theorem 4.4. *Let* X, Y, $\varphi : X \to \mathcal{F}_c(Y)$, $E \subset Y$ *and* $Z \subset X$ *be as in Theorem* 2.2. *For each* n, *let* $D_n = \{x \in X: \dim \varphi(x) = n\}$, *let* $Z_n = Z \cap D_n$, *and suppose that* $\dim_X Z_n < n - \dim \operatorname{conv}(\varphi(x) \cap E)$ *for all* $x \in Z_n$. *Then* φ *has a selection* f *which avoids* E *on* $\bigcup_n D_n$.

Proof. Let $A_n = \{x \in X: \dim \varphi(x) \leq n\}$. Then A_n is closed in X because φ is l.s.c., and $D_{n+1} = A_{n+1} \setminus A_n$. We will choose selections f_n for φ such that, for all n:

(a) $f_n(x) \notin E$ for all $x \in A_n$.

(b) $f_{n+1} | A_n = f_n | A_n$.

(c) $d(f_{n+1}(x), f_n(x)) \leq 2^{-n}$ for all $x \in X$.

That will suffice, for then (f_n) converges uniformly to a selection f for φ by (c), and this f avoids E on $\bigcup_n D_n$ by (a) and (b).

To construct the f_n, begin by letting f_0 be any selection for φ. Suppose we have $f_0, \ldots, f_n$, and let us choose f_{n+1}. Define $\theta : X \to \mathcal{F}_c(Y)$ by $\theta(x) = (\varphi(x) \cap B_{2^{-n}}(f_n(x)))^-$, and define $\psi : X \to \mathcal{F}_c(Y)$ by $\psi(x) = \theta(x)$ if $x \in X \setminus A_n$ and $\psi(x) = \{f_n(x)\}$ if $x \in A_n$. Then $\psi(x) \subset \theta(x) \subset \varphi(x)$ for all $x \in X$, and θ and ψ are both l.s.c. Also $\dim \psi(x) = \dim \varphi(x) = n+1$ for all $x \in D_{n+1}$ because $\psi(x)$ contains a non-empty, relatively open subset of the convex set $\varphi(x)$ for all $x \notin A_n$. Now let $Z' = \{x \in A_{n+1}: \psi(x) \cap E \neq \emptyset\}$; then $Z' \subset Z_{n+1}$ because $\psi(x) \subset \varphi(x)$ for all $x \in X$ and $\psi(x) \cap E = \emptyset$ for $x \in A_n$. Hence $\dim_X Z' \leq \dim_X Z_{n+1} < n + 1 - \dim \operatorname{conv}(\varphi(x) \cap E) \leq \dim \psi(x) - \dim \operatorname{conv}(\psi(x) \cap E)$ for all $x \in Z'$. Since $\dim \operatorname{conv}(\psi(x) \cap E) \leq \dim \varphi(x) \leq n+1$ for all $x \in A_{n+1}$, it follows from Theorem 4.1 that $\psi | A_{n+1}$ has an E-avoiding selection g. By Remark 1.1′, g extends to a selection f_{n+1} for ψ, and this f_{n+1} satisfies (a) (with n replaced by $n+1$), (b) and (c). $\square$

The following corollary is an immediate consequence of Theorem 4.4.

Corollary 4.5. *Theorem* 2.2 *remains true with* $\dim X$ *replaced by* $\dim_X Z$, *provided* $\dim \varphi(x) < \infty$ *for all* $x \in Z$.

5. The 0-dimensional case

In the special case where $\dim X = 0$, the results of the previous three sections can be improved and simplified. Our starting point is the following theorem, obtained from Theorem 1.2 by setting $n = -1$. An easy direct proof can be found in [6].

Theorem 5.1 [6, Theorem 2]. *Let X be paracompact, Y completely metrizable, $\varphi : X \to \mathscr{F}(Y)$ l.s.c., and suppose that* $\dim X = 0$. *Then φ has a selection.*

Our next result, a significant improvement over the 0-dimensional case of Theorem 2.2, follows immediately from Theorem 5.1.

Theorem 5.2. *Let X be paracompact, Y completely metrizable, $\varphi : X \to \mathscr{F}(Y)$ l.s.c., and $E \subset Y$ closed. Suppose that* $\dim X = 0$ *and that $\varphi(x)\backslash E \neq \emptyset$ for all $x \in X$. Then φ has an E-avoiding selection.*

Proof. Let $Y' = Y\backslash E$. Then Y' is open in Y and thus completely metrizable. Define $\psi : X \to \mathscr{F}(Y')$ by $\psi(x) = \varphi(x) \cap Y'$. Then ψ is also l.s.c. (because Y' is open in Y), so ψ has a selection f by Theorem 5.1. This f is an E-avoiding selection for φ. $\square$

Example 10.7 shows that, unlike Theorem 2.2, Theorem 5.2 becomes false if E is only assumed to be an F_σ in Y. See, however, Theorem 5.5 below.

The following result improves Theorem 3.1 when $\dim X = 0$. We denote $\{E: E \subset Y\}$ by $\mathscr{P}(Y)$.

Theorem 5.3. *Let X be paracompact, Y completely metrizable, and $\varphi : X \to \mathscr{F}(Y)$ l.s.c. Let $\psi : X \to \mathscr{P}(Y)$ have a closed graph in $X \times Y$, and suppose that $\dim X = 0$ and that $\varphi(x)\backslash\psi(x) \neq \emptyset$ for every $x \in X$. Then φ has a ψ-avoiding selection.*

Proof. Index X as (x_λ). By Theorem 5.1., there are selections f_λ for φ such that $f_\lambda(x_\lambda) \notin \psi(x_\lambda)$. Let $U_\lambda = \{x \in X: f_\lambda(x) \notin \psi(x)\}$; then (U_λ) is an open cover of X. Let (V_λ) be a disjoint open refinement of (U_λ), and define $f: X \to Y$ by $f(x) = f_\lambda(x)$ for $x \in V_\lambda$. This f is a ψ-avoiding selection for φ. $\square$

Theorem 5.3 generalizes Theorems 5.1 and 5.2. For an analogous generalization of Theorem 1.1, see Theorem 9.1.

The following result is analogous to Lemma 3.2. The fine topology is defined in Footnote 10.

Lemma 5.4. *Let X, Y, $\varphi : X \to \mathscr{F}(Y)$ and $\psi : X \to \mathscr{P}(Y)$ be as in Theorem 5.3, and suppose that $\varphi(x)\backslash\psi(x)$ is dense in $\varphi(x)$ for every $x \in X$. Let S be the space of selections for φ with the fine topology. Then S is a Baire space, and $\{f \in S: f \text{ avoids } \psi\}$ is open and dense in S.*

Proof. Almost identical to the proof of Lemma 3.2. $\square$

Our next result improves Theorem 3.3 when $\dim X = 0$.

Theorem 5.5. *Let X be paracompact, Y completely metrizable, and $\phi : X \to \mathscr{F}(Y)$ l.s.c.*

Let $\psi_i : X \to \mathcal{P}(Y)$ have a closed graph in $X \times Y$ for $i = 1, 2, \ldots,$ and suppose that $\dim X = 0$ *and that $\varphi(x) \backslash \psi_i(x)$ is dense in $\varphi(x)$ for all x and all i. Then φ has a selection which avoids every ψ_i.*

Proof. Immediate from Lemma 5.4. □

Theorem 5.7 below, which improves results in Section 4 when $\dim_X Z \leq 0$, generalizes Theorem 5.3 as well as Theorem 9.1 and [10, Theorem 1.1]. The proof of this result uses the following general lemma.

Lemma 5.6. *Let X be paracompact, Y a metric space, and $\varphi : X \to 2^Y$ l.s.c. Suppose that $\omega : X \to 2^Y$ has an open graph in $X \times Y$ and that $\omega(x) \cap \varphi(x) \neq \emptyset$ for all $x \in X$. Then there exists a $\theta : X \to 2^Y$ with open graph in $X \times Y$ such that:*

(a) *$\theta(x) \subset \omega(x)$ and $\theta(x) \cap \varphi(x) = \omega(x) \cap \varphi(x)$ for all $x \in X$.*

(b) *If f is a selection for θ, then there is a map $h : X \to \mathbb{R}^+$ such that $B_{h(x)}(f(x))$ intersects $\varphi(x)$ and its closure is a subset of $\omega(x)$ for all $x \in X$.*

Proof. Let $(U_\lambda, V_\lambda, t_\lambda)$ be the family of all triples such that U_λ is open in X, V_λ is open in Y, $t_\lambda > 0$, and whenever $(x, y) \in U_\lambda \times V_\lambda$ then $B_{t_\lambda}(y) \cap \varphi(x) \neq \emptyset$ and $B_{2t_\lambda}(y) \subset \omega(x)$. Let $\theta : X \to \mathcal{P}(Y)$ be the function whose graph is the open subset $\bigcup_\lambda (U_\lambda \times V_\lambda)$ of $X \times Y$.

It follows easily from our assumptions that θ satisfies (a). To see that it also satisfies (b), let f be a selection for θ. Let $U'_\lambda = \{x \in U_\lambda : f(x) \in V_\lambda\}$. Then (U'_λ) is an open cover of X, and thus has a locally finite partition of unity (p_λ) subordinated to it. Let $h(x) = \sum_\lambda p_\lambda(x) t_\lambda$. This h has the properties required by (b). □

The following theorem reduces to Theorem 5.3 when $Z = X$, to Theorem 9.1 when $Z = \emptyset$, and to [10, Theorem 1.1] when $\psi(x) = \emptyset$ for all $x \in X$. For the definition of $\dim_X Z$, see Footnote 11.

Theorem 5.7. *Let X be paracompact, Y a Banach space, and $\varphi : X \to \mathcal{F}(Y)$ l.s.c. Let $Z \subset X$ with* $\dim_X Z \leq 0$, *let $\psi : X \to \mathcal{P}(Y)$ have a closed graph in $X \times Y$, and suppose that $\varphi(x) \backslash \psi(x)$ is non-empty for all $x \in X$ and convex for all $x \in X \backslash Z$. Then φ has a ψ-avoiding selection.*

Proof. Define $\omega : X \to 2^Y$ by $\omega(x) = Y \backslash \psi(x)$. Then φ and ω satisfy the hypotheses of Lemma 5.6; let $\theta : X \to 2^Y$ be as in the conclusion of that lemma. Note that $\theta(x) \cap \varphi(x)$ is convex for all $x \in X \backslash Z$.

Write $X = (x_\lambda)$. It follows from [10, Theorem 1.1] that there are selections f_λ for φ such that $f_\lambda(x_\lambda) \in \theta(x_\lambda)$ for all λ. Let $U_\lambda = \{x \in X : f_\lambda(x) \in \theta(x)\}$. Then (U_λ) is an open cover of X, and thus has a locally finite open refinement (V_λ) such that $\bar{V}_\lambda \subset U_\lambda$ for all λ.

Define α, β, α_c, $\beta_c : X \to \mathscr{F}(Y)$ by $\alpha(x) = \{f_\lambda(x): x \in V_\lambda\}$, $\beta(x) = \{f_\lambda(x): x \in \bar{V}_\lambda\}$, $\alpha_c(x) = \operatorname{conv} \alpha(x)$ and $\beta_c(x) = \operatorname{conv} \beta(x)$. Then $\alpha(x) \subset \beta(x) \subset \theta(x) \cap \varphi(x)$ for all $x \in X$, and $\alpha_c(x) \subset \beta_c(x) \subset \theta(x) \cap \varphi(x)$ for all $x \in X \setminus Z$. Moreover, α and α_c are l.s.c. while β and β_c are u.s.c.[13] Let $W = \{x \in X: \beta_c(x) \subset \theta(x)\}$. Then $W \subset X \setminus Z$, and W is open in X because β_c is u.s.c., each $\beta_c(x)$ is compact, and the graph of θ is open. Let $A = X \setminus W$. Then A is closed in X and $A \subset Z$, so $\dim A \leqslant 0$.

By Theorem 5.1, $\alpha | A$ has a selection g, and this g extends to a selection f for α_c by Remark 1.1. Then $f(x) \in \alpha(x) \subset \theta(x)$ for all $x \in A$, and $f(x) \in \alpha_c(x) \subset \beta_c(x) \subset \theta(x)$ for all $x \in W$, so f is a selection for θ. Now let $h: X \to \mathbb{R}^+$ be as in Lemma 5.6(b), and define $\varphi' : X \to \mathscr{F}(Y)$ by $\varphi'(x) = (B_{h(x)}(f(x)) \cap \varphi(x))^-$. Then φ' is l.s.c. and $\varphi'(x)$ is convex for all $x \in X \setminus Z$, so φ' has a selection f by [10, Theorem 1.1]. Clearly f' is a selection for φ, and f' avoids ψ because $\varphi'(x) \subset \omega(x)$ by Lemma 5.6(b). $\square$

6. Selections for continuous φ

The significance of continuous φ for our purposes is embodied in the following theorem. As shown by Example 10.5, this theorem becomes false if φ is only assumed to be l.s.c.

Theorem 6.1. *Let Y be a metric space, and let $\mathscr{E} \subset \mathscr{F}(Y)$ satisfy the following conditions.*

(i) *Every $E \in \mathscr{E}$ is an AR.*[14]

(ii) *If $y \in E$ for some $E \in \mathscr{E}$, then $\{y\} \in \mathscr{E}$.*

(iii) *To every $\varepsilon > 0$ corresponds a $\delta = \delta(\varepsilon) > 0$ such that, if X is paracompact, if $\varphi : X \to \mathscr{E}$ is l.s.c., and if $h : X \to Y$ is a map with $d(h(x), \varphi(x)) < \delta$ for all $x \in X$, then φ has a selection f with $d(h(x), f(x)) < \varepsilon$ for all $x \in X$.*

Then $\mathscr{E}$ has the following properties:

(a) *If X is a paracompact, every continuous $\varphi : X \to \mathscr{E}$ has the selection extension property.*[15]

(b) *If X is any space, every continuous $\varphi : X \to \mathscr{E}$ has a selection f such that $f(x_1) = f(x_2)$ whenever $\varphi(x_1) = \varphi(x_2)$.*

The proof of Theorem 6.1 depends on the following general result, which is also of independent interest.

[13] See Footnote 12.

[14] A metrizable space E is an AR if and only if, whenever X is metrizable and $A \subset X$ is closed, then every map $g: A \to E$ extends to a map $f: X \to E$. If E is completely metrizable, this remains valid for paracompact X (see [1, Theorem 2]).

[15] See Remark 1.1. It should be observed that the argument given there, that the existence of selections for certain classes of l.s.c. functions φ implies that these φ actually have the selection extension property, is not valid for classes of *continuous* φ, since φ being continuous does not imply that φ_g is continuous.

Theorem 6.2. *Let X and Y be topological spaces, let $\mathscr{A}$ be a locally finite, closed cover of X, and let $\varphi: X \to 2^Y$ be such that $\varphi \mid A$ has the selection extension property for every $A \in \mathscr{A}$. Then φ has the selection extension property.*

Proof. That φ has a selection was proved (although not explicitly stated) in [7, Theorem 8.2], under a hypothesis on $\mathscr{A}$ somewhat weaker than being locally finite. Essentially the same proof shows that φ actually has the selection extension property. □

Proof of Theorem 6.1. (a) By Theorem 6.2, it will suffice to show that every $x_0 \in X$ has a neighborhood U such that $\varphi \mid C$ has the selection extension property for every closed (in X) $C \subset U$. Let ρ be the Hausdorff metric on $\mathscr{C}$, and define

$$U = \{x \in X: \rho(\varphi(x), \varphi(x_0)) < \delta(\delta(1))\},$$

where δ is as in 6.1(iii). Since φ is continuous, U is open in X.

We must show that, if $C \subset U$ is closed in X, and if $A \subset C$ is closed, then every selection g for $\varphi \mid A$ extends to a selection f for $\varphi \mid C$. Now $d(g(x), \varphi(x_0)) < \delta(\delta(1))$ for all $x \in A$, so (by 6.1(iii)) there exists a map $h: A \to \varphi(x_0)$ such that $d(h(x), g(x)) < \delta(1)$ for all $x \in A$. Since $\varphi(x_0)$ is an AR (by 6.1(i)), h extends to a map $h': C \to \varphi(x_0)$. Define $\psi: C \to \mathscr{C}$ by $\psi(x) = \{g(x)\}$ if $x \in A$ and $\psi(x) = \varphi(x)$ if $x \in C \backslash A$. Then ψ is l.s.c. and $d(h'(x), \psi(x)) < \delta(1)$ for all $x \in C$, so (by 6.1(iii) with X replaced by C) ψ has a selection f. This f is the required selection for $\varphi \mid C$ extending g.

(b) Let X' denote $\mathscr{C}$ with the Hausdorff metric, and define $\varphi': X' \to \mathscr{C}$ by $\varphi'(E) = E$ for every $E \in X'$. Since X' is paracompact and φ' is continuous, φ' has a selection f' by (a). If we now let $f = f' \circ \varphi$, then f is the required selection for φ. □

7. Proof of Theorem 1.4 and related results

In this section we apply Theorem 6.1 to subsets of a Banach space.

Lemma 7.1. *Let Y be a Banach space, let $S = \{y \in Y: \|y\| = 1\}$, and let $\mathscr{C}$ be the collection of all $E \in \mathscr{F}(S)$ such that $\|y\|^{-1} y \in E$ whenever $0 \neq y \in \operatorname{conv} E$. Then $\mathscr{C}$ satisfies condition* (iii) *of Theorem* 6.1.

Proof. Let $U = Y \backslash \{0\}$, and define the retraction $r: U \to S$ by $r(y) = \|y\|^{-1} y$. By our hypothesis, if $E \in \mathscr{C}$ then $r(U \cap \operatorname{conv} E) \subset E$, and this implies that $r(U \cap (\operatorname{conv} E)^-) \subset E$. It is easy to see that to every $\varepsilon > 0$ corresponds a $\gamma = \gamma(\varepsilon) > 0$ such that, if $y \in U$ and $d(y, S) < \gamma$, then $d(y, r(y)) < \varepsilon$; we may assume that $\gamma(\varepsilon) < \varepsilon$ and $\gamma(\varepsilon) < 1$.

We will show that condition (iii) of Theorem 6.1 is satisfied with $\delta = \delta(\varepsilon) = \frac{1}{2}\gamma(\frac{1}{2}\varepsilon)$. So let X be paracompact, let $\varphi: X \to \mathscr{C}$ be l.s.c., and let $h: X \to Y$ be continuous with $d(h(x), \varphi(x)) < \delta$ for all $x \in X$. Define $\psi: X \to \mathscr{F}_c(Y)$ by $\psi(x) =$

$(\mathrm{conv}\,\varphi(x)\cap B_\delta(h(x)))^-$. Then ψ is l.s.c. by [7, Propositions 2.3, 2.5 and 2.6)], so ψ has a selection g by Theorem 1.1. Suppose $x\in X$. Then $d(g(x),h(x))\leq\delta$, so $d(g(x),\varphi(x))<2\delta=\gamma(\frac{1}{2}\varepsilon)$, and hence $g(x)\in U$ and $d(rg(x),g(x))<\frac{1}{2}\varepsilon$. Since $g(x)\in U\cap(\mathrm{conv}\,\varphi(x))^-$, we have $rg(x)\in\varphi(x)$. Define $f=r\circ g$. Then f is a selection for φ, and $d(f(x),h(x))\leq d(f(x),g(x))+d(g(x),h(x))<\frac{1}{2}\varepsilon+\delta<\varepsilon$. □

Theorem 7.2. *Let Y be a Banach space, let $S=\{y\in Y\colon \|y\|=1\}$, and let $\mathscr{E}$ be the collection of all $E\in\mathscr{F}(S)$ such that E is an AR and $\|y\|^{-1}y\in E$ whenever $0\neq y\in\mathrm{conv}\,E$. Then*:

(a) *If X is paracompact, every continuous $\varphi:X\to\mathscr{E}$ has the selection extension property.*[16]

(b) *If X is any space, every continuous $\varphi:X\to\mathscr{E}$ has a selection f such that $f(x_1)=f(x_2)$ whenever $\varphi(x_1)=\varphi(x_2)$.*

Proof. By Lemma 7.1, the collection $\mathscr{E}$ satisfies the assumptions of Theorem 6.1. Hence the conclusion follows from Theorem 6.1. □

The following lemma will permit us to apply Theorem 7.2 to the proof of Theorem 1.4.

Lemma 7.3. *Let Y be a Banach space, and let $S=\{y\in Y\colon \|y\|=1\}$. Suppose that $D\subset Y$ is convex, that* $\dim D=\infty$ *and that $\|y\|^{-1}y\in D$ whenever $y\in D\backslash\{0\}$. Then $D\cap S$ is an AR.*

Proof. Since $D\cap S$ is a retract of $D\backslash\{0\}$, it will suffice to check that $D\backslash\{0\}$ is an AR. Now D, being convex, is an AR by Dugundji's Extension Theorem [2, Theorem 4.1], so the open subset $D\backslash\{0\}$ is an ANR. Since $\dim D=\infty$, Lemma 2.1 implies that $D\backslash\{0\}$ is C^n for all n. By another result of Dugundji [2, last paragraph on p. 362], every ANR which is C^n for all n must be an AR, so D is an AR. □

Proof of Theorem 1.4. Let $\varphi:X\to\mathscr{F}_c(Y)$ be as in Theorem 1.4. Let $S=\{y\in Y\colon \|y\|=1\}$, and define $\psi:X\to\mathscr{F}(S)$ by $\psi(x)=\varphi(x)\cap S$. It is easy to see that ψ is also continuous. Our assumptions on φ, together with Lemma 7.3, imply that ψ satisfies all the hypotheses of Theorem 7.2, so ψ has a selection f by Theorem 7.2(b). This f is clearly a 0-avoiding selection for φ. □

The above proof of Theorem 1.4 immediately implies that, just as in Theorem 7.2(b), the selection f for φ can be chosen so that $f(x_1)=f(x_2)$ whenever $\varphi(x_1)=\varphi(x_2)$. With rather more effort, one can also draw a conclusion which is analogous to Theorem 7.2(a); see Theorem 9.4.

[16] See Footnote 15.

8. Proof of Theorem 1.5

As already remarked after the statement of Theorem 1.5 in Section 1, we need only prove assertion (a).

With the notation of Theorem 1.5, define $\varphi : S^* \to \mathcal{F}(S)$ by $\varphi(u) = \{x \in S: u(x) = 0\}$. It will suffice to show that φ satisfies the hypotheses of Theorem 7.2, since our conclusion will then follow Theorem 7.2(b). Now if $u \in S^*$, then $\varphi(u)$ is the unit sphere of the infinite-dimensional Banach space $\{x \in B: u(x) = 0\}$, and is therefore an AR by a theorem of Dugundji [2, Theorem 6.2]. That $\|x\|^{-1}x \in \varphi(u)$ whenever $0 \neq x \in \operatorname{conv} \varphi(u)$ is clear. All that remains to be proved is that φ is continuous.

It will suffice to show that, if $u, v \in S^*$ with $\|u - v\| < \varepsilon < 1$, and if $x \in \varphi(u)$, then $\|y - x\| < 2\varepsilon$ for some $y \in \varphi(v)$. Now $|u(x) - v(x)| < \varepsilon$ (since $\|x\| = 1$), so $|v(x)| < \varepsilon$ because $u(x) = 0$. Let $E = \{x' \in B: \|x'\| < \varepsilon\}$. Since $\|v\| = 1$, the set $v(E)$ contains all scalars t with $|t| < \varepsilon$; in particular, $v(x') = v(x)$ for some $x' \in E$. Let $z = x - x'$, so $v(z) = 0$. Now $\|z - x\| = \|x'\| < \varepsilon$, so $|\,\|z\| - 1| < \varepsilon$. Hence $z \neq 0$, and we may define $y = \|z\|^{-1}z$. Then $\|y\| = 1$ and $v(y) = 0$, so $y \in \varphi(v)$. Also $\|y - z\| < \varepsilon$. Since $\|z - x\| < \varepsilon$, it follows that $\|y - x\| < 2\varepsilon$. □

9. Further results

The principal purpose of this section is to establish Theorem 9.3, which is used in the proofs of Theorem 9.4 and Example 10.4. We begin with the following result, which generalizes Theorem 1.1.

Theorem 9.1. *Let X be paracompact, Y a Banach space, and $\varphi : X \to \mathcal{F}(Y)$ l.s.c. Let $\psi : X \to 2^Y$ have an open graph in $X \times Y$, define $\theta(x) = \varphi(x) \cap \psi(x)$, and suppose that $\theta(x)$ is convex and on-empty for all $x \in X$. Then θ has a selection.*

Proof. Index X as (x_λ). For each λ, use Remark 1.1′ to pick a selection f_λ for φ such that $f_\lambda(x_\lambda) \in \psi(x_\lambda)$. Let $U_\lambda = \{x \in X: f_\lambda(x) \in \psi(x)\}$. Then $x_\lambda \in U_\lambda$ and U_λ is open in X. Let (p_λ) be a locally finite partition of unity on X subordinated to (U_λ), and define $f: X \to Y$ by $f(x) = \sum_\lambda p_\lambda(x) f_\lambda(x)$. This f is a selection for θ. □

We now introduce some notation. Let Y be a linear space. If $E \subset Y$, define $\hat{E} = \{\alpha y: y \in E, \alpha > 0\}$. Call $E \subset Y$ *radial* if $\hat{E} = E$. If $\varphi : X \to 2^Y$, define $\hat{\varphi} : X \to 2^Y$ by $\hat{\varphi}(x) = (\varphi(x))\hat{}$. The following lemma, whose routine verification is omitted, records several basic properties of these concepts.

Lemma 9.2. *Let Y be a normed linear space. Then*:

(a) *If $E \subset Y$ is convex, so is $\hat{E}$.*
(b) *If $E \subset Y$ is radial, so is $\bar{E}$.*
(c) *If $E, F \subset Y$, and if $E \cap \hat{F} \neq \emptyset$, then $\hat{E} \cap F \neq \emptyset$.*

(d) *If* $\varphi : X \to 2^Y$ *is l.s.c., then so is* $\hat{\varphi}$.

(e) *If* $\psi : X \to 2^Y$, *and if the graph of* ψ *is open in* $X \times Y$, *then so is the graph of* $\hat{\psi}$.

The next result depends on Theorem 9.1 and Lemma 9.2.

Theorem 9.3. *Let* X *be paracompact,* Y *a Banach space, and* $\varphi : X \to \mathscr{F}_c(Y)$ *l.s.c. Define* $\varphi^* : X \to \mathscr{F}_c(Y)$ *by* $\varphi^*(x) = (\hat{\varphi}(x))^-$. *Then* φ^* *is also l.s.c.,* $\varphi^*(x)$ *is radial for every* $x \in X$, *and if* φ^* *has an* 0-*avoiding selection then so does* φ.

Proof. Since φ is l.s.c., so is $\hat{\varphi}$ by Lemma 9.2(d), and hence so is φ^*. Since $\hat{\varphi}(x)$ is clearly radial, so is $\varphi^*(x)$ by Lemma 9.2(b).

Suppose that φ^* has an 0-avoiding selection g. Define $\psi : X \to 2^Y$ by $\psi(x) = \{y \in Y: d(y, g(x)) < \|g(x)\|\}$. Then $\psi(x)$ is convex for all $x \in X$, and hence so is $\hat{\psi}(x)$ by Lemma 9.2(a). Clearly $0 \notin \psi(x)$, so $0 \notin \hat{\psi}(x)$ for all $x \in X$. Since the graph of ψ is open in $X \times Y$, so is the graph of $\hat{\psi}$ by Lemma 9.2(e). Let $\theta(x) = \varphi(x) \cap \hat{\psi}(x)$; then $\theta(x)$ is convex and $0 \notin \theta(x)$ for all $x \in X$. Also $\psi(x)$ intersects $\hat{\varphi}(x)$, hence $\hat{\psi}(x)$ intersects $\varphi(x)$ by Lemma 9.2(c), and thus $\theta(x) \neq \emptyset$ for all $x \in X$. Hence θ has a selection f by Theorem 9.1, and this f is 0-avoiding selection for φ. □

We now use Theorems 9.3 and 7.2 to prove the following strengthening of Theorem 1.4 when X is paracompact.

Theorem 9.4. *For paracompact* X, *the conclusion of Theorem* 1.4 *can be strengthened as follows: If* $A \subset X$ *is closed, then every* 0-*avoiding selection* g *for* $\varphi | A$ *extends to a* 0-*avoiding selection for* φ.

Proof. Define $\varphi_g : X \to \mathscr{F}_c(Y)$ by $\varphi_g(x) = \varphi(x)$ if $x \notin A$ and $\varphi_g(x) = \{g(x)\}$ if $x \in A$. We want a 0-avoiding selection for φ_g; by Theorem 9.3, it will suffice to find a 0-avoiding selection for $\hat{\varphi}_g$.

Let $S = \{y \in Y: \|y\| = 1\}$, and define $\psi : X \to \mathscr{F}(S)$ by $\psi(x) = \varphi(x) \cap S$ for all $x \in X$. It is easy to see that ψ is continuous. Let $\mathscr{C} \subset \mathscr{F}(S)$ be as in Theorem 7.2; our assumptions on φ and Lemma 7.3 imply that $\psi(x) \in \mathscr{C}$ for every $x \in X$. Define $h : A \to S$ by $h(x) = \|g(x)\|^{-1} g(x)$. Then h is a selection for $\psi | A$, and hence h extends to a selection f for ψ by Theorem 7.2(a). This f is the required 0-avoiding selection for $\hat{\varphi}_g$. □

10. Examples

Our first example shows how Theorem 1.3 can become false if the strict inequality $\dim X < \varphi(x)$ is not satisfied.

Example 10.1. Let B be a Banach space, X a non-empty, compact, convex subset of B, and define the continuous $\varphi : X \to \mathscr{F}_c(B)$ by $\varphi(x) = X - x$ (where the minus sign is algebraic). Then φ does not have a 0-avoiding selection.

Proof. Suppose f were such a selection. Define the map $g : X \to X$ by $g(x) = f(x) + x$. Then g has no fixed point, violating the Schauder fixed-point theorem for X. □

We now apply Example 10.1 to the Hilbert cube $Q = \{x \in l_2 : |x_n| \leq 1/n \text{ for all } n\}$. Recall that Q is compact and convex, and that $\dim Q = \infty$.

Example 10.2. Let $Q \subset l_2$ be the Hilbert cube, and define $\varphi : Q \to \mathscr{F}_c(l_2)$ by $\varphi(x) = Q - x$. Then φ is continuous and $\dim \varphi(x) = \infty$ for all $x \in X$, but φ has no 0-avoiding selection.

Proof. Immediate from Example 10.1. □

Our next example shows that a very natural candidate for an infiite-dimensional analogue of Theorem 1.2 is false.[17] We call a collection $\mathscr{E}$ for subsets of Y *strongly equi-ANR* if, for every $y \in \bigcup \mathscr{E}$, every neighborhood V of y in Y contains a neighborhood W of y in Y such that $W \cap E$ is an AR for every $E \in \mathscr{E}$.

Example 10.3. Let $Y = l_2 \backslash \{0\}$, and define $\psi : Q \to \mathscr{F}(Y)$ by $\psi(x) = \varphi(x) \backslash \{0\}$, where Q and φ are as in Example 10.2. Then Y is completely metrizable, ψ is l.s.c., every $\psi(x)$ is an AR and $\{\psi(x) : x \in X\}$ is strongly equi-ANR, but ψ does not have a selection.

Proof. Since Y is an open subset of the complete metric space l_2, it is completely metrizable. Since φ is l.s.c., and $\psi(x) = \varphi(x) \cap Y$ with Y open in l_2, it follows that ψ is also l.s.c. Next, note that every $\psi(x)$ is homeomorphic to Q with a point removed. Since $Q \backslash \{(1/n)\}$ is convex and hence an AR by Dugundji's Extension Theorem [2], and since Q is homogeneous by a result of Keller [5], the sets $\psi(x)$ are all homeomorphic AR's. To see that $\{\psi(x) : x \in Q\}$ is strongly equi-ANR, suppose that $y \in Y$ and that V is a neighborhood of y in Y. Let $W \subset V$ be a neighborhood of y in Y which is a convex subset of l_2; then $W \cap \psi(x) = W \cap \varphi(x)$, so $W \cap \psi(x)$ is convex and therefore an AR. Finally, ψ has no selection because such a selection would be a 0-avoiding selection for φ, and that is impossible by Example 10.2. □

The following example shows that "continuous" cannot be weakened to "l.s.c." in Theorem 1.4.

Example 10.4. There exists an l.s.c. $\varphi^* : Q \to \mathscr{F}_c(l_2)$, such that $\dim \varphi^*(x) = \infty$ for all

[17] As indicated in Section 1, such an example was already obtained by Pixley in [12]. Our example is simpler than Pixley's, but our sets $\varphi(x)$—unlike Pixley's—are not compact.

$x \in Q$ and $\|y\|^{-1}y \in \varphi^*(x)$ whenever $0 \neq y \in \varphi^*(x)$ and $x \in X$, but φ^* has no 0-avoiding selection.

Proof. Let $\varphi: Q \to \mathscr{F}_c(l_2)$ be as in Example 10.2, and define $\varphi^*: Q \to \mathscr{F}_c(l_2)$ as in Theorem 9.3. This φ^* has all the required properties. $\square$

We next show that "continuous" cannot be weakened to "l.s.c." in Theorem 6.1.

Example 10.5. There exists a collection $\mathscr{C} \subset \mathscr{F}(l_2)$ satisfying conditions (i)–(iii) of Theorem 6.1, and an l.s.c. $\theta: Q \to \mathscr{C}$, such that θ has no selection.

Proof. Let $\varphi^*: Q \to \mathscr{F}_c(l_2)$ be as in Example 10.4. Let $S = \{y \in l_2: \|y\| = 1\}$ and define $\theta: Q \to \mathscr{F}(S)$ by $\theta(x) = \varphi^*(x) \cap S$ for $x \in Q$. Let $\mathscr{C} = \{\theta(x): x \in Q\} \cup \{\{y\}: y \in S\}$. Then $\mathscr{C}$ satisfies 6.1(i)–(iii) by Lemmas 7.1 and 7.3, θ is easily seen to be l.s.c., and θ has no selection because φ^* has no 0-avoiding selection. $\square$

The next example shows that the assumption that E is closed in Y cannot be dropped from Theorem 2.2, even when $\dim X = 0$ and $\text{card}(\varphi(x) \cap E) = 1$ for every $x \in X$.

Example 10.6. There exists a continuous $\varphi: X \to \mathscr{F}_c(\mathbb{R}^2)$, with X the Cantor set and $\varphi(x)$ an interval for each $x \in X$, and a set $E \subset \mathbb{R}^2$ with $\text{card}(\varphi(x) \cap E) = 1$ for all $x \in X$, such that φ has no E-avoiding selection.

Proof. Such an example is given in [9, Example 4.1], except that there the space X is an interval. However, exactly the same construction works when X is the Cantor set. $\square$

Our last example shows that Theorem 5.2 becomes false if E is only assumed to be an F_σ in Y.

Example 10.7. There exists a continuous $\varphi: X \to \mathscr{F}_c(\mathbb{R}^2)$, with $X = \{0\} \cup \{n^{-1}: n \in \omega\}$ and $\varphi(x)$ an interval for each $x \in X$, and an F_σ-set $E \subset \mathbb{R}^2$ with $\varphi(x) \backslash E \neq \emptyset$ for all $x \in X$, such that φ has no E-avoiding selection.

Proof. Let $\varphi(x) = \{x\} \times [0, 1]$, and let $E = (\{0\} \times (0, 1]) \cup ((X \backslash \{0\}) \times [0, 1))$. Clearly φ has no E-avoiding selection. $\square$

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